

EXPONENTIAL SUM ESTIMATE FOR SYSTEMS INCLUDING LINEAR POLYNOMIALS

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ABSTRACT. In his paper [3], W. M. Schmidt obtained an exponential sum estimate for systems of polynomials not including linear polynomials, which was then used to apply the Hardy-Littlewood circle method. We prove an analogous estimate for systems including linear polynomials.

1. INTRODUCTION

Let $\mathbf{u} = (\mathbf{u}_d, \dots, \mathbf{u}_1)$ be a system of polynomials in $\mathbb{Q}[x_1, \dots, x_n]$, where $\mathbf{u}_\ell = (u_{\ell,1}, \dots, u_{\ell,r_\ell})$ is the degree ℓ polynomials of \mathbf{u} ($1 \leq \ell \leq d$). We let $\mathbf{U} = (\mathbf{U}_d, \dots, \mathbf{U}_1)$ be the system of forms, where for each $1 \leq \ell \leq d$, $\mathbf{U}_\ell = (U_{\ell,1}, \dots, U_{\ell,r_\ell})$ and $U_{\ell,r}$ is the degree ℓ portion of $u_{\ell,r}$ ($1 \leq r \leq r_\ell$). Let us denote $\mathfrak{B}_0 = [0, 1]^n$. We define the following exponential sum associated to \mathbf{u} ,

$$(1.1) \quad S(\boldsymbol{\alpha}) = S(\mathbf{u}, \mathfrak{B}_0; \boldsymbol{\alpha}) := \sum_{\mathbf{x} \in P\mathfrak{B}_0 \cap \mathbb{Z}^n} e \left(\sum_{1 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \alpha_{\ell,r} \cdot u_{\ell,r}(\mathbf{x}) \right).$$

In his paper [3], W. M. Schmidt obtained an exponential sum estimate for $S(\boldsymbol{\alpha})$ when \mathbf{u} has integer coefficients, does not include linear polynomials, and satisfies certain properties. The estimate was then used in applying the Hardy-Littlewood circle method to obtain the asymptotic formula for the number of integer points of bounded height on the affine variety defined by \mathbf{u} . We refer the reader to [3] for more details on this important work. The work of Schmidt was found useful in the breakthrough of B. Cook and Á. Magyar [2], where they count the number of solutions whose coordinates are all prime to diophantine equations, and also in [4]. It makes sense for Schmidt in [3] to only consider systems without linear polynomials, because he is concerned with integer points and linear polynomials can be eliminated via substitution in this case. However, if one wants to apply the result of Schmidt for a coordinate dependent problem (where one can not eliminate linear polynomials by substitution), then it may be useful to have analogous exponential sum estimates for systems including linear polynomials, and this is what we achieve in this paper.

We need to introduce some notations before we can state our result. Let $1 < \ell \leq d$ and $r_\ell > 0$. We let $\mathbb{M}_\ell = \mathbb{M}_\ell(\mathbf{U}_\ell)$ be the affine variety in $(\mathbb{C}^n)^{\ell-1}$ associated to \mathbf{U}_ℓ , for which the definition we provide in (2.1) of Section 2. For $R_0 > 0$, we denote $z_{R_0}(\mathbb{M}_\ell)$ to be the number of integer points $(\mathbf{x}_1, \dots, \mathbf{x}_{\ell-1})$ on \mathbb{M}_ℓ such that

$$\max_{1 \leq i \leq \ell-1} \max_{1 \leq j \leq n} |x_{ij}| \leq R_0,$$

Date: Revised on July 29, 2016.

2010 Mathematics Subject Classification. 11L07, 11P55.

Key words and phrases. Hardy-Littlewood circle method, exponential sum estimate.

where $\mathbf{x}_i = (x_{i1}, \dots, x_{in})$ ($1 \leq i \leq \ell - 1$). We define $g_\ell(\mathbf{U}_\ell)$ to be the largest real number such that

$$(1.2) \quad z_P(\mathbb{M}_\ell) \ll P^{n(\ell-1)-g_\ell(\mathbf{U}_\ell)+\varepsilon}$$

holds for each $\varepsilon > 0$.

Let

$$\gamma_\ell = \frac{2^{\ell-1}(\ell-1)r_\ell}{g_\ell(\mathbf{U}_\ell)}$$

when $r_\ell > 0$ and $g_\ell(\mathbf{U}_\ell) > 0$. We let $\gamma_\ell = 0$ if $r_\ell = 0$, and let $\gamma_\ell = +\infty$ if $r_\ell > 0$ and $g_\ell(\mathbf{U}_\ell) = 0$.

These quantities are not defined for linear polynomials. When $\ell = 1$, following [2] we define $\mathcal{B}_1(\mathbf{u}_1)$ to be the minimum number of non-zero coefficients in a non-trivial linear combination

$$\lambda_1 U_1 + \dots + \lambda_{r_1} U_{r_1},$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{r_1}) \in \mathbb{Q}^{r_1} \setminus \{\mathbf{0}\}$. Clearly $\mathcal{B}_1(\mathbf{u}_1) > 0$ if and only if the linear forms U_1, \dots, U_{r_1} are linearly independent over \mathbb{Q} . If $r_1 = 0$ then we let $\mathcal{B}_1(\mathbf{u}_1) = +\infty$.

The following theorem is the main result of this paper.

Theorem 1.1. *Suppose \mathbf{u} has coefficients in \mathbb{Z} , and that*

$$\mathcal{B}_1(\mathbf{u}_1) > 2r_1 \left(\max \left\{ 4(r_1 + 1) \left(\sum_{j=2}^d 4^{j-2} \gamma_j \right), \frac{1}{4(R+1)} \right\} \right)^{-1}.$$

Let

$$0 < \Omega < \min \left\{ \frac{1}{8r_1 + 9} \left(\sum_{j=2}^d 4^{j-2} \gamma_j \right)^{-1}, \left(\frac{1}{2(R+1)} + \sum_{j=2}^d 4^{j-2} \gamma_j \right)^{-1} \right\}.$$

Let $0 < \Delta \leq 1$, and let P be sufficiently large with respect to $n, d, r_d, \dots, r_1, \Delta, \Omega$, and \mathbf{u} . Then one of the following two alternatives must hold:

(i) $|S(\boldsymbol{\alpha})| \leq P^{n-\Delta\Omega}.$

(ii) There exists $q \in \mathbb{N}$ such that

$$q \leq P^\Delta \quad \text{and} \quad \|q\boldsymbol{\alpha}_\ell\| \leq P^{-\ell+\Delta} \quad (1 \leq \ell \leq d).$$

In Section 2, we also prove a lemma on estimating the quantity known as the singular integral, which comes up in the Hardy-Littlewood circle method. We use \ll and \gg to denote Vinogradov's well-known notation. We also use the notation $e(x)$ to denote $e^{2\pi i x}$. For $q \in \mathbb{N}$, we use the numbers from $\{0, 1, \dots, q-1\}$ to represent the residue classes of $\mathbb{Z}/q\mathbb{Z}$.

2. PROOF OF THEOREM 1.1

First we present the following lemma from [2].

Lemma 2.1. [2, Lemma 3] *Let $\mathbf{G} = (G_1, \dots, G_{r'})$ be a system of linear forms in $\mathbb{Q}[x_1, \dots, x_n]$. Given any $1 \leq j \leq n$, we have*

$$\mathcal{B}_1(\mathbf{G}|_{x_j=0}) \geq \mathcal{B}_1(\mathbf{G}) - 1.$$

Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{x}_j = (x_{j,1}, \dots, x_{j,n})$ for $j \geq 1$. Given a function $G(\mathbf{x})$, we define

$$\Gamma_{\ell,G}(\mathbf{x}_1, \dots, \mathbf{x}_\ell) = \sum_{t_1=0}^1 \dots \sum_{t_\ell=0}^1 (-1)^{t_1+\dots+t_\ell} G(t_1\mathbf{x}_1 + \dots + t_\ell\mathbf{x}_\ell).$$

Then it follows that $\Gamma_{\ell,G}$ is symmetric in its ℓ arguments, and that $\Gamma_{\ell,G}(\mathbf{x}_1, \dots, \mathbf{x}_{\ell-1}, \mathbf{0}) = 0$ [3, Section 11]. We also have that if G is a form of degree d and $\ell > d > 0$, then $\Gamma_{\ell,G} = 0$ [3, Lemma 11.2].

For $\alpha \in \mathbb{R}$, let $\|\alpha\|$ denote the distance from α to the closest integer. Let $\boldsymbol{\alpha} = (\alpha_d, \dots, \alpha_1) \in \mathbb{R}^R$, where $R = r_1 + \dots + r_d$ and $\boldsymbol{\alpha}_\ell = (\alpha_{\ell,1}, \dots, \alpha_{\ell,r_\ell}) \in \mathbb{R}^{r_\ell}$ ($1 \leq \ell \leq d$). We define

$$\|\boldsymbol{\alpha}\| = \max_{\substack{1 \leq \ell \leq d \\ 1 \leq r \leq r_\ell}} \|\alpha_{\ell,r}\| \quad \text{and} \quad |\boldsymbol{\alpha}| = \max_{\substack{1 \leq \ell \leq d \\ 1 \leq r \leq r_\ell}} |\alpha_{\ell,r}|.$$

Let $\mathbf{u} = (\mathbf{u}_d, \dots, \mathbf{u}_1)$ and $\mathbf{U} = (\mathbf{U}_d, \dots, \mathbf{U}_1)$ be as in Section 1. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis vectors of \mathbb{C}^n . Let $1 < \ell \leq d$. We define $\mathbb{M}_\ell = \mathbb{M}_\ell(\mathbf{U}_\ell)$ to be the set of $(\ell-1)$ -tuples $(\mathbf{x}_1, \dots, \mathbf{x}_{\ell-1}) \in (\mathbb{C}^n)^{\ell-1}$ for which the matrix

$$(2.1) \quad [m_{ir}] = [\Gamma_{\ell,U_{\ell,r}}(\mathbf{x}_1, \dots, \mathbf{x}_{\ell-1}, \mathbf{e}_i)] \quad (1 \leq r \leq r_\ell, 1 \leq i \leq n)$$

has rank strictly less than r_ℓ .

Lemma 2.2 below is the inhomogeneous polynomials version of [3, Lemma 15.1], and it is obtained by essentially the same proof. We refer the reader to [3, Section 9] and ‘Remark on inhomogeneous polynomials’ in [3, pp. 262] for further explanation. We remark that the implicit constants may depend on \mathbf{u} here, and not only on \mathbf{U} . We also note that [3, Lemma 15.1] is for systems without linear polynomials in contrast to Lemma 2.2 below. However, it is clear from the proof of [3, Lemma 15.1] that the lemma is not affected with the presence of linear polynomials.

Lemma 2.2. [3, Lemma 15.1] *Suppose \mathbf{u} has coefficients in \mathbb{Z} . Let $Q > 0$ and $\varepsilon > 0$. Let $2 \leq \ell \leq d$ with $r_\ell > 0$. Let P be sufficiently large with respect to d and r_d, \dots, r_1 . If $\ell = d$, then let $\theta = 0$ and $q = 1$. On the other hand, if $2 \leq \ell < d$, then suppose $0 \leq \theta < 1/4$ and that there is $q \in \mathbb{N}$ with*

$$q \leq P^\theta \quad \text{and} \quad \|q\boldsymbol{\alpha}_j\| \leq P^{\theta-j} \quad (\ell < j \leq d).$$

Let $S(\boldsymbol{\alpha})$ be the sum associated to \mathbf{u} as in (1.1). Given $\eta > 0$ with $\eta + 4\theta \leq 1$, one of the following three alternatives must hold:

$$(i) \quad |S(\boldsymbol{\alpha})| \leq P^{n-Q}.$$

(ii) *There exists $n_0 \in \mathbb{N}$ such that*

$$n_0 \ll P^{r_\ell(\ell-1)\eta} \quad \text{and} \quad \|qn_0\boldsymbol{\alpha}_\ell\| \ll P^{-\ell+4\theta+r_\ell(\ell-1)\eta}.$$

$$(iii) \quad z_{R_0}(\mathbb{M}_\ell) \gg R_0^{(\ell-1)n-2^{\ell-1}(Q/\eta)-\varepsilon} \quad \text{holds with } R_0 = P^\eta.$$

The implicit constants depend at most on $n, d, r_d, \dots, r_1, \eta, \varepsilon$, and \mathbf{u} .

We are left to deal with the case $\ell = 1$ in Lemma 2.2. Given $\boldsymbol{\epsilon} \in (\mathbb{N} \cup \{0\})^n$ and sufficiently differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, put

$$\partial^\epsilon f = \frac{\partial^{\epsilon_1+\dots+\epsilon_n} f}{\partial x_1^{\epsilon_1} \dots \partial x_n^{\epsilon_n}}.$$

Let $\mathcal{C}^n(\mathbb{R}^n)$ be the set of n -th continuously differentiable functions defined on \mathbb{R}^n .

For $\epsilon \in \{0, 1\}^n$, we define $\bar{\epsilon} = (1, 1, \dots, 1) - \epsilon$. Given $\mathbf{t} = (t_1, \dots, t_n)$, we let \mathbf{t}_ϵ be the vector whose i -th coordinate equals zero if $\epsilon_i = 0$ and equals t_i if $\epsilon_i = 1$. Similarly, given $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{Z}^n$, we let $\mathbf{N}_{\bar{\epsilon}}$ be the vector whose i -th coordinate equals N_i if $\epsilon_i = 0$ and equals zero if $\epsilon_i = 1$. The following is a generalization of the partial summation formula obtained by applying induction on the dimension.

Lemma 2.3. [1, Lemma 2.1] *Let $\varrho : \mathbb{Z}^n \rightarrow \mathbb{C}$ be a function, and let*

$$T_\varrho(\mathbf{t}) = \sum_{0 \leq x_1 \leq t_1} \dots \sum_{0 \leq x_n \leq t_n} \varrho(\mathbf{x}).$$

Then for any $f \in \mathcal{C}^n(\mathbb{R}^n)$ we have

$$(2.2) \quad \sum_{\substack{0 \leq x_i \leq N_i \\ (1 \leq i \leq n)}} f(\mathbf{x}) \varrho(\mathbf{x}) = \sum_{\epsilon \in \{0, 1\}^n} \left(\prod_{1 \leq i \leq n} (-1)^{\epsilon_i} N_i^{\epsilon_i - 1} \right) \cdot \int_{[0, N_1]} \dots \int_{[0, N_n]} \partial^\epsilon f(\mathbf{N}_{\bar{\epsilon}} + \mathbf{t}_\epsilon) T_\varrho(\mathbf{N}_{\bar{\epsilon}} + \mathbf{t}_\epsilon) dt_n \dots dt_1.$$

Let us use the following notations. For $\mathbf{a} = (a_1, \dots, a_{r_1}) \in (\mathbb{Z}/q\mathbb{Z})^{r_1}$, we let

$$\mathfrak{M}_{\mathbf{a}, q}(C) = \{\alpha_1 \in [0, 1]^{r_1} : \max_{1 \leq r \leq r_1} |\alpha_{1,r} - a_r/q| \leq P^{C-1}\},$$

$$\mathfrak{M}(C) = \bigcup_{\substack{\gcd(\mathbf{a}, q)=1 \\ \mathbf{a} \in (\mathbb{Z}/q\mathbb{Z})^{r_1} \\ 1 \leq q \leq P^C}} \mathfrak{M}_{\mathbf{a}, q}(C),$$

and

$$\mathfrak{m}(C) = [0, 1]^{r_1} \setminus \mathfrak{M}(C).$$

We also let

$$\mathfrak{N}_{a, q}(C) = \{\alpha \in [0, 1] : |\alpha - a/q| \leq P^{C-1}\},$$

$$\mathfrak{N}(C) = \bigcup_{\substack{\gcd(a, q)=1 \\ 0 \leq a < q \\ 1 \leq q \leq P^C}} \mathfrak{N}_{a, q}(C),$$

and

$$\mathfrak{n}(C) = [0, 1] \setminus \mathfrak{N}(C).$$

With the use of Lemma 2.3, we obtain the following result when $r_1 > 0$.

Lemma 2.4. *Suppose \mathbf{u} has coefficients in \mathbb{Z} and that $r_1 > 0$. Let $0 < \theta_0 < 1$, and suppose there exists $q \in \mathbb{N}$ with*

$$q \leq P^{\theta_0} \quad \text{and} \quad \|q\alpha_j\| \leq P^{\theta_0-j} \quad (1 < j \leq d).$$

Let $S(\alpha)$ be the sum associated to \mathbf{u} as in (1.1). Let $\varepsilon_0 > 0$ be sufficiently small. Let $Q > 0$ and $0 < Q_0 < 1$ be two real numbers such that

$$\theta_0 < \frac{Q_0/2 - \varepsilon_0}{2r_1}$$

and

$$Q < \mathcal{B}_1(\mathbf{u}_1) \left(\frac{Q_0/2 - \varepsilon_0}{r_1} - 2\theta_0 \right).$$

Suppose P is sufficiently large with respect to $d, n, r_d, \dots, r_1, \varepsilon_0, \theta_0, Q_0, Q$, and \mathbf{u} . Then one of the following two alternatives must hold:

(i) $|S(\boldsymbol{\alpha})| \leq P^{n-Q}.$

(ii) There exists $n_0 \in \mathbb{N}$ such that

$$n_0 \leq P^{Q_0} \text{ and } \|n_0 \boldsymbol{\alpha}_1\| \leq P^{Q_0-1}.$$

Proof. If the alternative (ii) holds then we are done. Thus let us assume it is not the case. Suppose $\boldsymbol{\alpha}_1 \in \mathfrak{M}(Q_0/2)$. Then for some $1 \leq q' \leq P^{Q_0/2}$ and $a_1, \dots, a_{r_1} \in \mathbb{Z}$, we have

$$\max_{1 \leq r \leq r_1} |\alpha_{1,r} - a_r/q'| \leq P^{(Q_0/2)-1}$$

from which it follows that

$$\|q' \boldsymbol{\alpha}_1\| \leq P^{Q_0-1},$$

and this is a contradiction. Therefore, we have $\boldsymbol{\alpha}_1 \in \mathfrak{m}(Q_0/2)$.

For simplicity we denote $B = \mathcal{B}_1(\mathbf{u}_1)$ and $Q'_0 = Q_0/2$. Let us also denote

$$\sum_{r=1}^{r_1} \alpha_{1,r} \cdot U_{1,r}(\mathbf{x}) = \gamma_1 x_1 + \dots + \gamma_n x_n.$$

We let \widetilde{M}_1 be the $n \times r_1$ matrix, where its (j, r) -th entry is the x_j coefficient of $U_{1,r}(\mathbf{x})$. Since this matrix has full rank (because $B > 0$), let us take an invertible $r_1 \times r_1$ minor, which we assume without loss of generality to be the first r_1 rows of \widetilde{M}_1 , and denote it M_1 .

Suppose $\gamma_1, \dots, \gamma_{r_1} \in \mathfrak{N}(C')$ for some $C' > 0$. Then there exist integers a_1, \dots, a_{r_1} and q_1, \dots, q_{r_1} such that $\gcd(a_r, q_r) = 1$, $0 < q_r \leq P^{C'}$, and $|\gamma_r - a_r/q_r| \leq P^{C'}/P$ ($1 \leq r \leq r_1$). Let us define

$$\begin{bmatrix} a'_1/q' \\ \vdots \\ a'_{r_1}/q' \end{bmatrix} = M_1^{-1} \cdot \begin{bmatrix} a_1/q_1 \\ \vdots \\ a_{r_1}/q_{r_1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \beta'_1 \\ \vdots \\ \beta'_{r_1} \end{bmatrix} = M_1^{-1} \cdot \begin{bmatrix} \gamma_1 - a_1/q_1 \\ \vdots \\ \gamma_{r_1} - a_{r_1}/q_{r_1} \end{bmatrix}.$$

It is easy to deduce that we have

$$q' \leq P^{r_1 C' + \varepsilon_0} \quad \text{and} \quad |\beta'_r| \leq \frac{P^{r_1 C' + \varepsilon_0}}{P} \quad (1 \leq r \leq r_1)$$

when P is sufficiently large with respect to the coefficients of \mathbf{U}_1 . Since $\alpha_{1,r} = \frac{a'_r}{q'} + \beta'_r$ ($1 \leq r \leq r_1$), we see that $\boldsymbol{\alpha}_1 \in \mathfrak{M}(r_1 C' + \varepsilon_0)$. However, since $\boldsymbol{\alpha}_1 \in \mathfrak{m}(Q'_0)$, it follows from this argument that at least one of $\gamma_1, \dots, \gamma_{r_1}$ is in $\mathfrak{n}((Q'_0 - \varepsilon_0)/r_1)$. Without loss of generality, we suppose that $\gamma_1 \in \mathfrak{n}((Q'_0 - \varepsilon_0)/r_1)$.

Let \widetilde{M}_2 be the matrix obtained by removing the first row of \widetilde{M}_1 . If $B - 1 > 0$, then we know that \widetilde{M}_2 has full rank. Let us take an invertible $r_1 \times r_1$ minor, which we assume without loss of generality to be the first r_1 rows of \widetilde{M}_2 , and denote it M_2 . By the same argument as above, we obtain without loss of generality that $\gamma_2 \in \mathfrak{n}((Q'_0 - \varepsilon_0)/r_1)$. In fact we can repeat the argument B times, and obtain that $\gamma_1, \gamma_2, \dots, \gamma_B \in \mathfrak{n}((Q'_0 - \varepsilon_0)/r_1)$.

Since $q \leq P^{\theta_0} \leq P^{(Q'_0 - \varepsilon_0)/r_1}$, it then follows that

$$(2.3) \quad \frac{P^{(Q'_0 - \varepsilon_0)/r_1}}{P} < \|q\gamma_i\| \quad (1 \leq i \leq B).$$

For each $2 \leq \ell \leq d, 1 \leq r \leq r_\ell$, let $a_{\ell,r} \in \mathbb{Z}$ and $\beta_{\ell,r} \in \mathbb{R}$ be such that

$$(2.4) \quad \alpha_{\ell,r} - a_{\ell,r}/q = \beta_{\ell,r} \quad \text{and} \quad |\beta_{\ell,r}| \leq P^{\theta_0 - \ell}.$$

We then consider

$$(2.5) \quad |S(\boldsymbol{\alpha})| = \left| \sum_{\substack{0 \leq k_i < q \\ (1 \leq i \leq n)}} \sum_{\substack{\mathbf{x} \in [0, P]^n \\ x_i \equiv k_i \pmod{q} \\ (1 \leq i \leq n)}} e \left(\sum_{1 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \alpha_{\ell,r} \cdot u_{\ell,r}(\mathbf{x}) \right) \right| \\ \leq q^n \max_{\substack{0 \leq k_i < q \\ (1 \leq i \leq n)}} \left| \sum_{\substack{0 \leq y_i \leq (P - k_i)/q \\ (1 \leq i \leq n)}} e \left(\sum_{1 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \alpha_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y} + \mathbf{k}) \right) \right|.$$

Let us denote

$$f(\mathbf{y}) = e \left(\sum_{2 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \beta_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y} + \mathbf{k}) \right).$$

Using the fact that $e(m) = 1$ for $m \in \mathbb{Z}$, we can simplify the above inequality (2.5) further,

$$|S(\boldsymbol{\alpha})| \leq q^n \max_{\substack{0 \leq k_i < q \\ (1 \leq i \leq n)}} \left| \sum_{\substack{0 \leq y_i \leq (P - k_i)/q \\ (1 \leq i \leq n)}} e \left(\sum_{2 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \beta_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y} + \mathbf{k}) + \sum_{1 \leq r \leq r_1} \alpha_{1,r} \cdot U_{1,r}(q\mathbf{y}) \right) \right| \\ \leq q^n \max_{\substack{0 \leq k_i < q \\ (1 \leq i \leq n)}} \sum_{\substack{0 \leq y_i \leq (P - k_i)/q \\ (B < i \leq n)}} \left| \sum_{\substack{0 \leq y_i \leq (P - k_i)/q \\ (1 \leq i \leq B)}} f(\mathbf{y}) e \left(\sum_{1 \leq i \leq B} q\gamma_i y_i \right) \right|.$$

Let $0 \leq y_i \leq (P - k_i)/q$ ($B < i \leq n$). Given $\boldsymbol{\epsilon} \in \{0, 1\}^B$, let $\left(\frac{(P - \mathbf{k})}{q}\right)_{\bar{\boldsymbol{\epsilon}}}$ be the vector whose i -th coordinate, for $1 \leq i \leq B$, equals $(P - k_i)/q$ if $\epsilon_i = 0$ and equals zero if $\epsilon_i = 1$, and for $B < i \leq n$, equals y_i . We also let $\mathbf{t}_{\boldsymbol{\epsilon}}$ be the vector whose i -th coordinate, for $1 \leq i \leq B$, equals 0 if $\epsilon_i = 0$ and equals t_i if $\epsilon_i = 1$, and for $B < i \leq n$, equals zero.

We prove that given $\boldsymbol{\epsilon} \in \{0, 1\}^B$ and $0 \leq t_i \leq (P - k_i)/q$ ($1 \leq i \leq B$), we have

$$(2.6) \quad \frac{\partial^{\epsilon_1 + \dots + \epsilon_B} f}{\partial y_1^{\epsilon_1} \dots \partial y_B^{\epsilon_B}} \Big|_{\mathbf{y} = \left(\frac{(P - \mathbf{k})}{q}\right)_{\bar{\boldsymbol{\epsilon}}} + \mathbf{t}_{\boldsymbol{\epsilon}}} \ll q^{\epsilon_1 + \dots + \epsilon_B} P^{(\theta_0 - 1)(\epsilon_1 + \dots + \epsilon_B)},$$

where the implicit constant is independent of $k_1, \dots, k_n, y_{B+1}, \dots, y_n$, and \mathbf{t} . In order to prove this statement, without loss of generality suppose $\epsilon_i = 1$ for $1 \leq i \leq E$ and $\epsilon_i = 0$ for $E < i \leq B$. The statement is trivial if $\epsilon_i = 0$ for all $1 \leq i \leq B$. Let $i_1 < \dots < i_m \leq E$. First

note when $m \leq d$, we have from (2.4) that

$$\begin{aligned}
 (2.7) \quad & \frac{\partial^m}{\partial y_{i_1} \dots \partial y_{i_m}} \left(\sum_{2 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \beta_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y} + \mathbf{k}) \right) \Big|_{\mathbf{y} = \left(\frac{P-\mathbf{k}}{q}\right)_{\bar{\epsilon}} + \mathbf{t}_\epsilon} \\
 & \ll q^m \sum_{\max\{2,m\} \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \beta_{\ell,r} P^{\ell-m} \\
 & \ll q^m P^{\theta_0-m},
 \end{aligned}$$

and when $m > d$,

$$(2.8) \quad \frac{\partial^m}{\partial y_{i_1} \dots \partial y_{i_m}} \left(\sum_{2 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \beta_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y} + \mathbf{k}) \right) = 0.$$

Thus we have

$$\begin{aligned}
 (2.9) \quad & \frac{\partial^E f}{\partial y_1 \dots \partial y_E} \Big|_{\mathbf{y} = \left(\frac{P-\mathbf{k}}{q}\right)_{\bar{\epsilon}} + \mathbf{t}_\epsilon} \ll \max_{\substack{m_1 + \dots + m_j = E \\ 1 \leq m_i \leq d \\ (1 \leq i \leq j)}} q^E P^{j\theta_0-E} \\
 & \ll q^E P^{(\theta_0-1)E},
 \end{aligned}$$

from which we can deduce (2.6). We now prepare to apply Lemma 2.3. Let $0 \leq t_i \leq (P - k_i)/q$ ($1 \leq i \leq B$). It follows from (2.3) that

$$(2.10) \quad \left| \sum_{0 \leq y_i \leq t_i} e(q\gamma_i y_i) \right| \ll \min \left\{ t_i + 1, \|q\gamma_i\|^{-1} \right\} \leq P^{1-(Q'_0-\varepsilon_0)/r_1} \quad (1 \leq i \leq B).$$

Then for $\epsilon \in \{0, 1\}^B$, we have by (2.6) and (2.10) that

$$\begin{aligned}
 (2.11) \quad & \int_{[0, (P-k_1)/q]} \dots \int_{[0, (P-k_B)/q]} \partial^\epsilon f \left(\left(\frac{P-\mathbf{k}}{q} \right)_{\bar{\epsilon}} + \mathbf{t}_\epsilon \right) \cdot \\
 & \sum_{\substack{0 \leq y_i \leq (P-k_i)/q \\ \epsilon_i=0}} \sum_{\substack{0 \leq y_i \leq t_i \\ \epsilon_i=1}} e \left(\sum_{1 \leq i \leq B} q\gamma_i y_i \right) dt_B \dots dt_1 \\
 & \ll q^{\epsilon_1 + \dots + \epsilon_B} P^{(\theta_0-1)(\epsilon_1 + \dots + \epsilon_B)} \left(\prod_{1 \leq i \leq B} \frac{P-k_i}{q} \right) \cdot P^{B-B(Q'_0-\varepsilon_0)/r_1}.
 \end{aligned}$$

Therefore, by Lemma 2.3 and (2.11) we obtain for any $0 \leq y_i \leq (P - k_i)/q$ ($B < i \leq n$),

$$\begin{aligned}
 (2.12) \quad & \left| \sum_{\substack{0 \leq y_i \leq (P-k_i)/q \\ (1 \leq i \leq B)}} f(\mathbf{y}) e \left(\sum_{1 \leq i \leq B} q\gamma_i y_i \right) \right| \\
 & \ll \sum_{\epsilon \in \{0,1\}^B} \left(\prod_{1 \leq i \leq B} \left(\frac{P-k_i}{q} \right)^{\epsilon_i-1} \right) q^{\epsilon_1 + \dots + \epsilon_B} \cdot \\
 & P^{(\theta_0-1)(\epsilon_1 + \dots + \epsilon_B)} \left(\prod_{1 \leq i \leq B} \frac{P-k_i}{q} \right) P^{B-B(Q'_0-\varepsilon_0)/r_1} \\
 & \ll P^{B\theta_0} P^{B-B(Q'_0-\varepsilon_0)/r_1}.
 \end{aligned}$$

Thus we obtain that (2.5) is bounded by

$$\begin{aligned} |S(\boldsymbol{\alpha})| &\ll q^n \left(\frac{P}{q}\right)^{n-B} P^{B\theta_0} P^{B-B(Q'_0-\varepsilon_0)/r_1} \\ &\leq q^B P^{n+B\theta_0-B(Q'_0-\varepsilon_0)/r_1} \\ &\leq P^{n+2B\theta_0-B(Q'_0-\varepsilon_0)/r_1}. \end{aligned}$$

Since we chose Q to satisfy

$$Q < B \left(\frac{Q_0/2 - \varepsilon_0}{r_1} - 2\theta_0 \right),$$

it follows that we are in alternative (i) as long as P is sufficiently large with respect to \mathbf{u} , d , n , r_d, \dots, r_1 , and Q . \square

Let $1 < \ell \leq d$ and $r_\ell > 0$. We define $g_\ell(\mathbf{U}_\ell)$ to be the largest real number such that

$$(2.13) \quad z_P(\mathbb{M}_\ell) \ll P^{n(\ell-1)-g_\ell(\mathbf{U}_\ell)+\varepsilon}$$

holds for each $\varepsilon > 0$. Let

$$\gamma_\ell = \frac{2^{\ell-1}(\ell-1)r_\ell}{g_\ell(\mathbf{U}_\ell)}$$

when $r_\ell > 0$ and $g_\ell(\mathbf{U}_\ell) > 0$. We let $\gamma_\ell = 0$ if $r_\ell = 0$, and let $\gamma_\ell = +\infty$ if $r_\ell > 0$ and $g_\ell(\mathbf{U}_\ell) = 0$. For ℓ with $r_\ell > 0$, we also define

$$(2.14) \quad \gamma'_\ell = \frac{2^{\ell-1}}{g_\ell(\mathbf{U}_\ell)} = \frac{\gamma_\ell}{(\ell-1)r_\ell}.$$

From Lemma 2.2, we obtain the following corollary which is the inhomogeneous polynomials version of [3, pp.276, Corollary], and it is obtained by essentially the same proof.

Corollary 2.5. [3, pp.276, Corollary] *Suppose \mathbf{u} has coefficients in \mathbb{Z} . Let $Q > 0$ and $\varepsilon > 0$. Let $2 \leq \ell \leq d$ with $r_\ell > 0$. Let P be sufficiently large with respect to d and r_d, \dots, r_1 . If $\ell = d$, then let $\theta = 0$ and $q = 1$. On the other hand, if $2 \leq \ell < d$, then suppose $0 \leq \theta < 1/4$ and that there is $q \in \mathbb{N}$ with*

$$q \leq P^\theta \quad \text{and} \quad \|q\boldsymbol{\alpha}_j\| \leq P^{\theta-j} \quad (\ell < j \leq d).$$

Let $S(\boldsymbol{\alpha})$ be the sum associated to \mathbf{u} as in (1.1). Suppose

$$4\theta + Q\gamma'_\ell < 1.$$

Then one of the following two alternatives must hold:

$$(i) \quad |S(\boldsymbol{\alpha})| \leq P^{n-Q}.$$

(ii) There exists $n_0 \in \mathbb{N}$ such that

$$n_0 \ll P^{Q\gamma_\ell+\varepsilon} \quad \text{and} \quad \|n_0 q \boldsymbol{\alpha}_\ell\| \ll P^{-\ell+4\theta+Q\gamma_\ell+\varepsilon}.$$

The implicit constants depend at most on $n, d, r_d, \dots, r_1, \varepsilon$, and \mathbf{u} .

The above corollary does not deal with the case $\ell = 1$, and we take care of this in the following lemma.

Lemma 2.6. [3, Lemma 15.2] Suppose \mathbf{u} has coefficients in \mathbb{Z} , and that

$$\mathcal{B}_1(\mathbf{u}_1) > 2r_1 \left(\max \left\{ 4(r_1 + 1) \left(\sum_{j=2}^d 4^{j-2} \gamma_j \right), \frac{1}{4(R+1)} \right\} \right)^{-1}.$$

Let $\varepsilon > 0$ be sufficiently small. Let $Q > 0$ satisfy

$$Q(8r_1 + 8) \left(\sum_{j=2}^d 4^{j-2} \gamma_j \right) < 1 \quad \text{and} \quad \frac{Q}{2(R+1)} < 1.$$

Let $S(\boldsymbol{\alpha})$ be the sum associated to \mathbf{u} as in (1.1). Suppose P is sufficiently large with respect to $d, n, r_d, \dots, r_1, \varepsilon, Q$, and \mathbf{u} . Then one of the following two alternatives must hold:

(i) $|S(\boldsymbol{\alpha})| \leq P^{n-Q}$.

(ii) There exist $n_1, n_2, \dots, n_d \in \mathbb{N}$ such that

$$\begin{aligned} n_\ell &\ll P^{Q\gamma_\ell + \varepsilon} \quad \text{and} \quad \|n_d \dots n_\ell \boldsymbol{\alpha}_\ell\| \ll P^{-\ell + Q(\sum_{j=\ell}^d 4^{j-\ell} \gamma_j) + \varepsilon} \quad (2 \leq \ell \leq d), \\ n_1 &\leq P^{M_0 Q} \quad \text{and} \quad \|n_1 \boldsymbol{\alpha}_1\| \leq P^{-1 + M_0 Q}, \end{aligned}$$

where

$$M_0 = \max \left\{ 8(r_1 + 1) \left(\sum_{j=2}^d 4^{j-2} \gamma_j \right), \frac{1}{2(R+1)} \right\}.$$

The implicit constants depend at most on $n, d, r_d, \dots, r_1, \varepsilon$, and \mathbf{u} .

Proof. We begin by proceeding as in the proof of [3, Lemma 15.2]. Suppose we have

$$|S(\boldsymbol{\alpha})| > P^{n-Q}.$$

Let $\varepsilon_d > 0$ be sufficiently small. Since $Q\gamma'_d < 1$, by Corollary 2.5 there exists $n_d \in \mathbb{N}$ with

$$n_d \ll P^{Q\gamma_d + \varepsilon_d} \quad \text{and} \quad \|n_d \boldsymbol{\alpha}_d\| \ll P^{-d + Q\gamma_d + \varepsilon_d}.$$

Suppose now that $r_{d-1} > 0$. Since $4Q\gamma_d + Q\gamma'_{d-1} < 1$, we can apply Corollary 2.5 again with $\ell = d-1$, $\theta = Q\gamma_d + 2\varepsilon_d$, and $q = n_d$. Note we have by our assumption on Q that $\theta < 1/4$. Let $\varepsilon_{d-1} > 0$ be sufficiently small. Thus there exists $n_{d-1} \in \mathbb{N}$ with

$$(2.15) \quad n_{d-1} \ll P^{Q\gamma_{d-1} + \varepsilon_{d-1}} \quad \text{and} \quad \|n_d n_{d-1} \boldsymbol{\alpha}_{d-1}\| \ll P^{-(d-1) + 4Q\gamma_d + 8\varepsilon_d + Q\gamma_{d-1} + \varepsilon_{d-1}}.$$

In the case $r_{d-1} = 0$, we have $\gamma_{d-1} = 0$ and obtain (2.15) trivially with $n_{d-1} = 1$. It is clear we can continue in this manner. By repeating the argument, we ultimately obtain that there exist $n_2, \dots, n_d \in \mathbb{N}$ such that

$$n_\ell \ll P^{Q\gamma_\ell + \varepsilon} \quad \text{and} \quad \|n_d \dots n_\ell \boldsymbol{\alpha}_\ell\| \ll P^{-\ell + Q(\sum_{j=\ell}^d 4^{j-\ell} \gamma_j) + \varepsilon} \quad (2 \leq \ell \leq d).$$

If $r_1 = 0$, then we are done trivially with $n_1 = 1$. Let $r_1 > 0$. We now apply Lemma 2.4 with

$$\theta_0 = \left(\sum_{j=2}^d 4^{j-2} \gamma_j \right) Q + d\varepsilon < 1,$$

where $\varepsilon > 0$ is sufficiently small,

$$Q_0/2 = \max \left\{ 4(r_1 + 1) \left(\sum_{j=2}^d 4^{j-2} \gamma_j \right) Q, \frac{Q}{4(R+1)} \right\} < \frac{1}{2},$$

and

$$q = (n_d \dots n_2) \leq P^{\theta_0},$$

where the last inequality holds for P sufficiently large. Let $\varepsilon_0 > 0$ be sufficiently small. With these choices of θ_0 and Q_0 , we have

$$2\theta_0 < (Q_0/2 - \varepsilon_0)/(2r_1) < (Q_0/2 - \varepsilon_0)/r_1.$$

With our assumption on $\mathcal{B}_1(\mathbf{u}_1)$, it is clear that we have

$$Q < \mathcal{B}_1(\mathbf{u}_1) \left(\frac{Q_0/2 - \varepsilon_0}{2r_1} \right) < \mathcal{B}_1(\mathbf{u}_1) \left(\frac{Q_0/2 - \varepsilon_0}{r_1} - 2\theta_0 \right).$$

Therefore, it follows by Lemma 2.4 that there exists $n_1 \in \mathbb{N}$ such that

$$n_1 \leq P^{Q_0} \quad \text{and} \quad \|n_1 \boldsymbol{\alpha}_1\| \leq P^{Q_0-1}.$$

□

We are now in position to prove our main result.

Proof of Theorem 1.1. By the hypotheses, we know that

$$(8r_1 + 8)\Delta\Omega (\gamma_2 + 4\gamma_3 + 4^2\gamma_4 + \dots + 4^{d-2}\gamma_d) < 1,$$

$$\frac{\Delta\Omega}{2(R+1)} < 1,$$

and

$$(2.16) \quad \Omega (\gamma_2 + 4\gamma_3 + 4^2\gamma_4 + \dots + 4^{d-2}\gamma_d) + \Omega M_0 < 1,$$

where

$$M_0 = \max \left\{ 8(r_1 + 1) \left(\sum_{j=2}^d 4^{j-2}\gamma_j \right), \frac{1}{2(R+1)} \right\}$$

as in the statement of Lemma 2.6.

Let

$$(2.17) \quad \varepsilon'_0 = \frac{1}{2\Omega} (1 - \Omega (\gamma_2 + 4\gamma_3 + 4^2\gamma_4 + \dots + 4^{d-2}\gamma_d) - \Omega M_0).$$

We apply Lemma 2.6 with $Q = \Delta\Omega$. If the alternative (i) of Lemma 2.6 holds then we are done. Let us suppose we have the alternative (ii) of Lemma 2.6. Then for P sufficiently large, we have

$$q := n_d \dots n_2 n_1 \leq P^{\Delta\Omega(\sum_{j=2}^d 4^{j-2}\gamma_j) + \Delta\Omega M_0 + \Delta\Omega \varepsilon'_0},$$

and

$$\|q\boldsymbol{\alpha}_\ell\| \leq P^{-\ell + \Delta\Omega(\sum_{j=2}^d 4^{j-2}\gamma_j) + \Delta\Omega M_0 + \Delta\Omega \varepsilon'_0} \quad (1 \leq \ell \leq d).$$

Since

$$\Omega (\gamma_2 + 4\gamma_3 + 4^2\gamma_4 + \dots + 4^{d-2}\gamma_d) + \Omega M_0 + \Omega \varepsilon'_0 < 1,$$

we obtain our result. □

We prove the following lemma which becomes useful in some applications of the Hardy-Littlewood circle method. The proof is based on that of [3, Lemma 8.1]. Let

$$\mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) = \int_{\mathbf{v} \in \mathfrak{B}_0} e \left(\sum_{\ell=1}^d \sum_{r=1}^{r_\ell} \tau_{\ell,r} \cdot U_{\ell,r}(\mathbf{v}) \right) d\mathbf{v}.$$

Lemma 2.7. [3, Lemma 8.1] *Suppose \mathbf{u} has coefficients in \mathbb{Z} , and that $\mathcal{B}_1(\mathbf{u}_1)$ is sufficiently large with respect to r_d, \dots, r_1 , and d . Furthermore, suppose $\gamma_2, \dots, \gamma_d$ are sufficiently small with respect to r_d, \dots, r_1 , and d . Then we have*

$$(2.18) \quad \mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) \ll \min(1, |\boldsymbol{\tau}|^{-R-1}),$$

where the implicit constant depends at most on n, d, r_d, \dots, r_1 , and \mathbf{U} .

Proof. Given $\mathbf{a} = (\mathbf{a}_d, \dots, \mathbf{a}_1) \in (\mathbb{Z}/q\mathbb{Z})^R$, where $\mathbf{a}_\ell = (a_{\ell,1}, \dots, a_{\ell,r_\ell}) \in (\mathbb{Z}/q\mathbb{Z})^{r_\ell}$ ($1 \leq \ell \leq d$) and $\gcd(\mathbf{a}, q) = 1$, let us define

$$\widetilde{\mathfrak{M}}_{\mathbf{a},q}((R+2)^{-1}) = \{\boldsymbol{\alpha} \in [0,1)^R : \max_{1 \leq r \leq r_\ell} |q\alpha_{\ell,r} - a_{\ell,r}| \leq P^{(R+2)^{-1}}/P^\ell \ (1 \leq \ell \leq d)\},$$

and let

$$\widetilde{\mathfrak{M}} = \bigcup_{q \leq P^{(R+2)^{-1}}} \bigcup_{\substack{\mathbf{a} \in (\mathbb{Z}/q\mathbb{Z})^R \\ \gcd(\mathbf{a}, q) = 1}} \widetilde{\mathfrak{M}}_{\mathbf{a},q}((R+2)^{-1}).$$

Note the boxes $\widetilde{\mathfrak{M}}_{\mathbf{a},q}((R+2)^{-1})$ with $q \leq P^{(R+2)^{-1}}$, $\mathbf{a} \in (\mathbb{Z}/q\mathbb{Z})^R$, and $\gcd(\mathbf{a}, q) = 1$ are disjoint when P is sufficiently large.

Suppose $|\boldsymbol{\tau}| > 2$. Let $P\mathbf{v} = \mathbf{v}'$ so that we have

$$\mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) = \frac{1}{P^n} \int_{P\mathfrak{B}_0} e \left(\sum_{\ell=1}^d \sum_{r=1}^{r_\ell} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{v}') \right) d\mathbf{v}',$$

where

$$(2.19) \quad \beta_{\ell,r} = \frac{\tau_{\ell,r}}{P^\ell} \ (1 \leq \ell \leq d, 1 \leq r \leq r_\ell).$$

Let $P = |\boldsymbol{\tau}|^{R+2}$, and consider the exponential sum

$$S(\boldsymbol{\beta}) = \sum_{\mathbf{x} \in P\mathfrak{B}_0 \cap \mathbb{Z}^n} e \left(\sum_{\ell=1}^d \sum_{r=1}^{r_\ell} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{x}) \right).$$

Then $\boldsymbol{\beta}$ lies on the boundary of the box $\widetilde{\mathfrak{M}}_{\mathbf{0},1}((R+2)^{-1})$. Thus for $|\boldsymbol{\tau}|$ sufficiently large, $\boldsymbol{\beta}$ lies on the boundary of the set $\widetilde{\mathfrak{M}}$, which is precisely the set considered in the alternative (ii) of Proposition 1.1 with $\Delta = (R+2)^{-1}$. Consequently, $\boldsymbol{\beta}$ also lies on the boundary of $[0,1)^R \setminus \widetilde{\mathfrak{M}}$. Since $|S(\boldsymbol{\alpha})|$ is a continuous function, we obtain via Theorem 1.1 (with $\Omega = R+1$) that

$$(2.20) \quad |S(\boldsymbol{\beta})| \leq P^{n-(R+2)^{-1}\Omega} = P^n |\boldsymbol{\tau}|^{-\Omega} = P^n |\boldsymbol{\tau}|^{-R-1}.$$

Note with the hypothesis of this lemma, we have

$$\begin{aligned} & \min \left\{ \frac{1}{8r_1+9} \left(\sum_{j=2}^d 4^{j-2} \gamma_j \right)^{-1}, \left(\frac{1}{2(R+1)} + \sum_{j=2}^d 4^{j-2} \gamma_j \right)^{-1} \right\} \\ &= \left(\frac{1}{2(R+1)} + \sum_{j=2}^d 4^{j-2} \gamma_j \right)^{-1} \\ &> R+1, \end{aligned}$$

which justifies our application of Theorem 1.1 with $\Omega = R+1$.

We also have

$$\begin{aligned}
(2.21) \quad & S(\boldsymbol{\beta}) - \int_{P\mathfrak{B}_0} e \left(\sum_{\ell=1}^d \sum_{r=1}^{r_\ell} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{v}') \right) d\mathbf{v}' \\
&= \sum_{\mathbf{x} \in [0,P]^n} \int_{x_1}^{x_1+1} \cdots \int_{x_n}^{x_n+1} e \left(\sum_{\ell=1}^d \sum_{r=1}^{r_\ell} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{x}) \right) - e \left(\sum_{\ell=1}^d \sum_{r=1}^{r_\ell} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{v}') \right) d\mathbf{v}' \\
&+ O(P^{n-1}) \\
&\ll P^n \frac{|\boldsymbol{\tau}|}{P} + O(P^{n-1}) \\
&\ll P^{n-1} |\boldsymbol{\tau}|,
\end{aligned}$$

where we applied the mean value theorem and (2.19) to obtain the second last inequality. Therefore, it follows that

$$S(\boldsymbol{\beta}) = P^n \mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) + O(P^{n-1} |\boldsymbol{\tau}|).$$

It is then easy to deduce from (2.20) that

$$\mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) \ll \min\{1, |\boldsymbol{\tau}|^{-R-1}\}.$$

□

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